Fast Computation of Chebyshev-Harmonic Fourier Moments

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***Abstract*—The orthogonal invariant descriptors are among the best region based shape descriptors which are used in many image processing and pattern recognition applications. The Chebyshev Harmonic Fourier Moments (CHFMs) are one of such invariant descriptor. They, however, suffer from high time complexity and numerical instability at high orders of moment. In this paper, we propose a fast method based on the recursive computation of radial kernel function of CHFMs which not only reduces time complexity but also improves their numerical stability.**

***Index Terms*—Chebyshev harmonic Fourier moments, recursive method, numerical stability**

I. Introduction

Orthogonal Rotation Invariant Moments (ORIMs) find a wide variety of applications and imaging science is one of the significant application areas. The ORIMs are distinguished from the other sets of moments by certain invariance properties which play the fundamental role in image understanding, symmetry detection and estimation and information retrieval. ORIMs possess the property of being invariant to rotation and can be made invariant to translation and scale after geometric transformations [1], [2]. Jacobi-Fourier Moments (JFMs) are a generic class of ORIMs recently introduced by Ping *et al.* [3] and further investigated by Hoang and Tabbone [4]. They provide a wide range of ORIMs whose radial kernel functions are polynomials. The widely used Zernike Moments (ZMs) [5], pseudo-Zernike Moments (PZMs) [6], orthogonal Fourier-Mellin moments OFMMs [7], and Chebyshev-Fourier moments [8] are special cases of JFMs. Ping *et al.* [8] introduced CHFMs and observed it to possess better noise sensitivity and reconstruction capability as compared to OFMMs. CHFMs have found several applications in image processing, pattern analysis and computer vision. However, they are less popular due to their low image reconstruction capabilities and numerical instability at low order moments as compared to ZMs, PZMs, and OFMMs. The computations of CHFMs involve factorial terms, which are computation intensive. Upneja and Singh [9] have proposed fast computation of JFMs of which CHFMs are a special case. However, when the method [9] is applied on the CHFMs, the number of arithmetic operations involved in the coefficients of the recursive relation is very high. Another method uses 8-way symmetry for the computation of the radial function $R_p(r)$ involved in CHFMs [10]. In this paper, we present a recursion based fast algorithm which reduces the time complexity of the CHFMs. The fast algorithm is based on the recursive computation of the radial and angular kernel functions of the moments. The proposed recursive method not only reduces the time complexity of moment computation but also enhances numerical stability of high order moments which is reflected in the lower values of image reconstruction error. The numerical stability is enhanced due to the fact that the proposed algorithm does not involve the direct computation of the factorial terms of large integers which appear in the radial polynomial $R_p(r)$.

The rest of the paper is organised as follows. An overview of CHFMs and its computational framework for digital images is presented in Section 2. A fast computational approach for the radial polynomials and angular functions is developed in Section 3. Detail experiments are conducted in Section 4 analyzing the time complexity and numerical stability. Section 5 concludes the paper.

II. Chebyshev Harmonic Fourier Moments (CHFMs)

CHFMs of order p and repetition q with $p \geq 0$ and $|q| \geq 0$ are defined in polar form as [8]:

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$$M_{pq} = \frac{1}{2\pi} \int_0^{2\pi} \int_0^1 f(r,\theta) V_{pq}^*(r,\theta)\, r dr d\theta \quad (1)$$

where p is a non-negative integer and q is an integer.

The function $V_{pq}^*(x,y)$ is the complex conjugate of the CHFMs basis function $V_{pq}(x,y)$ defined by:

$$V_{pq}(x,y) = R_p(r) e^{jq\theta} \quad (2)$$

where $r = \sqrt{x^2 + y^2}$.

The radial part of the basis function is:

$$R_p(r) = \sqrt{\frac{8}{\pi}} \left(\frac{1-r}{r}\right)^{1/4} \sum_{k=0}^{\lfloor p/2 \rfloor} (-1)^k \frac{(p-k)!}{k!\,(p-2k)!} \times \left(2(2r-1)\right)^{p-2k} \quad (3)$$

The orthogonal property for radial kernel is given as:

$$\int_0^1 R_p(r) R_k(r)\, r dr = \delta_{pk} \quad (4)$$

The orthogonality of basis function is given as:

$$\int_0^{2\pi} \int_0^1 V_{pq}(r,\theta) V_{p'q'}^*(r,\theta)\, r dr d\theta = 2\pi\, \delta_{pp'} \delta_{qq'} \quad (5)$$

For $p=p_{max}$, $q=q_{max}$, the total number of CHFMs is $(1+p_{\max})(1+2q_{\max})$.

In digital image processing, the image function $f(r,\theta)$ is discrete and defined in a rectangular domain with the pixel locations identified by the row and column arrangement. Let (i,k) be a pixel, the index i denotes the row position and k the column, with $i, k = 0, 1, \ldots, N-1$, where the resolution of the image is $N \times N$ pixels. The top left corner of the rectangular domain represents the origin $(0,0)$ of the image. We map the pixel location (i,k) into the coordinates (x_i, y_k) within the unit disk using the following transformation:

$$x_i = \frac{2i+1-N}{D},\ y_k = \frac{2k+1-N}{D},\ i,k = 0,1,\ldots,N-1 \quad (6)$$

where:

$$D = \begin{cases} N & \textit{for inscribed circular disk contained in the square image} \\ N\sqrt{2} & \textit{for outer circular disk containing the whole square image} \end{cases} \quad (7)$$

The coordinate (x_i, y_k) represents the center of the (i,k) pixel grid with the two opposite vertices defined by $\left[x_i - \frac{\Delta x}{2}, x_i + \frac{\Delta x}{2}\right] \times \left[y_k - \frac{\Delta y}{2}, y_k + \frac{\Delta y}{2}\right]$ where Δx and Δy represent the horizontal and vertical separation between the centers of two pixels which are expressed as:

$$\Delta x = \Delta y = \frac{2}{D} \quad (8)$$

The CHFMs can now be described in the Cartesian coordinates and their discrete formulation can be facilitated by converting (1) into Cartesian system defined by:

$$M_{pq} = \frac{1}{2\pi} \iint_{x_i^2+y_k^2 \le 1} f(x,y) V_{pq}^*(x,y)\, dx dy \quad (9)$$

Equation (9) can be derived from (1) after replacing $r = \sqrt{x^2+y^2}$ and θ by $\tan^{-1}(y/x)$. The discrete implementation of (9) assumes the form:

$$M_{pq} = \frac{1}{2\pi} \sum_{i=0}^{N-1} \sum_{k=0}^{N-1} f(x_i, y_k) \iint_{x_i^2+y_k^2 \le 1} V_{pq}^*(x,y)\, dx dy \quad (10)$$

It is difficult to derive an analytical solution to the double integration on the r.h.s of (10), therefore, normally a zeroth order approximation is considered for its evaluation. This leads to:

$$M_{pq} = \frac{4}{2\pi D^2} \sum_{\substack{i=0 \\ x_i^2+y_k^2 \le 1}}^{N-1} \sum_{k=0}^{N-1} f(x_i, y_k) V_{pq}^*(x_i, y_k) \quad (11)$$

Suppose that moments of all orders $p \le p_{\max}$ and repetition $q \le q_{\max}$ are given, then the image is reconstructed as follows:

$$\hat{f}(x_i, y_k) = \sum_{p=0}^{p_{\max}} \sum_{q=-q_{\max}}^{q_{\max}} M_{pq} V_{pq}(x_i, y_k),\ i, k = 0, 1, \ldots, N-1 \quad (12)$$

The image reconstruction error ε is defined by:

$$\varepsilon = \frac{\sum_{i=0}^{N-1} \sum_{k=0}^{N-1} \left(f(x_i,y_k) - \hat{f}(x_i,y_k)\right)^2}{\sum_{i=0}^{N-1} \sum_{k=0}^{N-1} f^2(x_i, y_k)} \quad (13)$$

III. Fast Computation of CHFMs

It is clear from (11) that the computation of M_{pq} involves the computation of the kernel function $V_{pq}^*(x_i, y_k)$ at N^2 locations. The computation of $V_{pq}^*(x_i, y_k)$ involves the computation of the radial polynomial $R_p(r)$ and the angular kernel function $e^{-jq\theta}$ both of which require heavy computational load. The order of time complexity of a polynomial $R_p(r)$ is $O(p)$. When all CHFMs upto a maximum order $p_{\max}$ are computed then the time complexity is $O(p_{\max}^2)$. The time complexity of the angular function $e^{-jq\theta}$ is $O(q_{\max})$ which is also very high because it involves the computation of the trigonometric functions $\cos(q\theta)$ and $\sin(q\theta)$ which are computation intensive. We propose a fast method based on recursion to evaluate $R_p(r)$. The proposed recursive method reduces the time complexity of a polynomial from $O(p)$ to $O(1)$ and time complexity of all CHFMs from $O(p_{\max}^2)$ to $O(p_{\max})$. Also, the angular functions are computed using recursion without making use of trigonometric functions [11].

A. Fast Computation of Radial Functions

We propose the following recursive relation to compute the radial functions $R_p(r)$ expressed by (3), which can be written as:

$$R_p(r)=\sqrt{\frac{8}{\pi}}\left(\frac{1-r}{r}\right)^{1/4}(-1)^p C_p(r) \quad (14)$$

where:

$$C_0(r)=1,\ C_1(r)=2(1-2r) \quad (15)$$

$$C_p(r)=C_1(r)C_{(p-1)}(r)-C_{(p-2)}(r),\quad p=2,3,\ldots,p_{\max} \quad (16)$$

These relations can also be derived from our work in [9] for the fast computation of the JFMs of which CHFMs is a special case. However, the formulation in [9] requires much more arithmetic operations to compute $C_p(r),\ p=0,1,\ldots,p-1,$ than that given in the proposed approach. It is clear that the order of the time complexity of the radial polynomials has been reduced to $O(1)$ from $O(p)$. Also, the recursive formulation does not involve the direct use of the factorial terms.

B. Fast Computation of Angular Functions

The angular function $e^{-jq\theta}=\cos(q\theta)-j\sin(q\theta)$, can be computed recursively using a method developed by Singh and Walia [11]. At a given pixel (i,k), the functions $\cos(\theta)$ and $\sin(\theta)$ are computed by:

$$r_{ik}=\sqrt{x_i^2+y_k^2},\ a=\cos\theta=\frac{x_i}{r_{ik}},\ b=\sin\theta=\frac{y_k}{r_{ik}} \quad (17)$$

For $q>1$, the following recursive expressions are used:

$$\begin{aligned}\cos(q\theta)&=a\cos((q-1)\theta)-b\sin((q-1)\theta)\\ \sin(q\theta)&=a\sin((q-1)\theta)+b\cos((q-1)\theta)\end{aligned} \quad (18)$$

with $\cos(0)=1$ and $\sin(0)=0$. The values of $\cos(m\theta)$, and $\sin(m\theta)$ are saved in two tables.

IV. Experimental Analysis

The proposed method for the computation of the CHFMs is compared with the conventional implementation. The comparison is performed for the following algorithms.

Algorithm A: This is the direct implementation of the CHFMs without recursion.

Algorithm B: This is the recursive algorithm given in [10] for the computation of JFMs. We have computed CHFMs using the relationship between JFMs and CHFMs.

Algorithm C: This is the proposed algorithm which uses recursion of the radial and angular functions.

These algorithms are implemented in Visual C++ 6.0 under Microsoft Windows environment on a PC with 3.0 GHz CPU and 2 GB RAM.

Since the speed of the algorithms doesn't depend on the image content, we use a 256×256 pixels image Lena for the experiments as shown in Fig. 1. The CPU elapse time versus the order and repetition of moments are plotted in Fig. 2. We have plotted the graphs for $p_{\max}=q_{\max}=5$ through 50 for the purpose of time analysis. It is shown that the direct implementation of CHFMs (Algorithm A) is very slow as compared to the proposed algorithm (Algorithms C). Algorithm B which gives comparable results with the proposed approach uses the recursive technique for the computation of JFMs. CHFMs are computed using the relationship between JFMs and CHFMs. However, the formulation is the recursive approach for JFMs but it requires very high arithmetic operations for the computation of CHFMs from JFMs. The proposed algorithm provides significant reduction in CPU elapse time for CHFMs of all orders and repetitions. The trend of CPU elapse time with respect to image size is plotted in Fig. 3. This image is resized to various scales for performing experiments when comparison in speed is performed on various scales of the image. Again, the qualitative trend of the CPU elapse time is similar to that obtained in Fig. 2. This is very useful for real time applications and applications involving large databases or on devices with low computation power.

Figure 1. Gray scale *Lena* image of 256×256 pixels resolution.

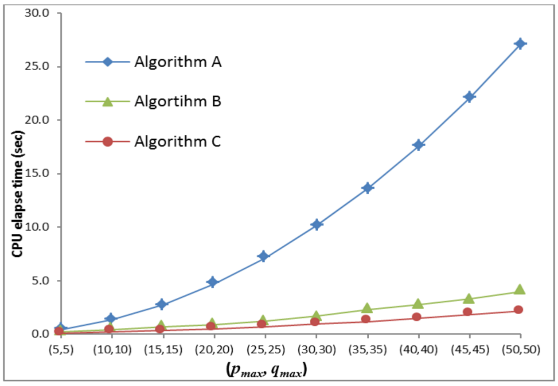


Figure 2. CPU elapse time (sec) for the computation of CHFMs for Algorithm A, Algorithm B and Algorithm C for Lena image with resolution 256×256 pixels and for various orders and repetitions of moment from $p_{max}=q_{max}=5$ through $p_{max}=q_{max}=50$.

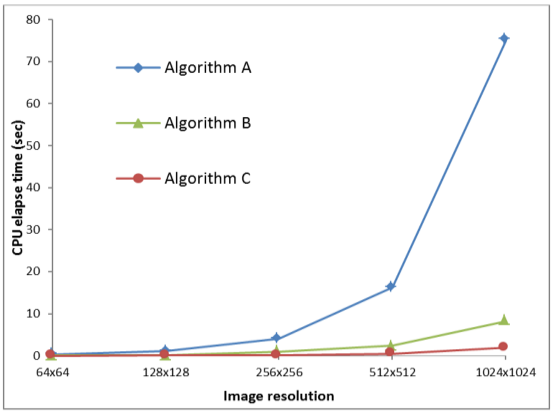


Figure 3. CPU elapse time (sec) for the computation of CHFMs for Algorithm A, Algorithm B and Algorithm C at fixed order and repetition $p_{max}=q_{max}=20$ and for various resolutions from 64×64 pixels through 1024×1024 pixels.

The direct method becomes numerically unstable at much lower values of p_{max} and q_{max} than the proposed recursive method. For the direct method, after reaching a minimum value of ε, the reconstruction error ε starts increasing at a very high rate. On the other hand the increase in ε after reaching its minimum value is very small w.r.t. p_{max} and q_{max} for recursive method. This demonstrates the stable behavior of the recursive method. It is also observed that the recursive method provides lower values of ε than the direct method.

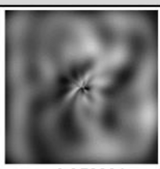
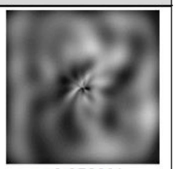
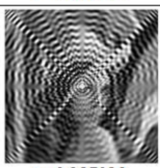
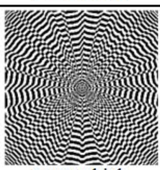
$p_{max} = q_{max}$	Direct method	Recursive method
10	 ε =0.079901	 ε=0.079901
20	 ε =0.043162	 ε =0.043162
30	 ε=0.032303	 ε =0.032303
40	 ε =0.040237	 ε =0.040234
50	 ε =0.327029	 ε =0.054968
60	 ε =very high	 ε =0.105155
70		 ε =0.201801

Figure 4. Reconstructed Lena image of 256×256 pixels using various CHFMs for different values of $p_{max} = q_{max}$.

The direct computation of CHFMs is numerically unstable which is reflected in the form of distortion in reconstructed images. Theoretically, higher values of order and repetition of CHFMs must provide reconstructed images closer to the original image. This characteristic is practically not achievable because of the finite precision arithmetic used in the digital computers. It is shown that the major cause of numerical instability is due to factorial terms involved in the polynomial function [9]. The proposed recursive computation of CHFMs polynomials not only reduces time complexity, but it also improves numerical stability. The improvement in numerical stability is reflected in the quality of reconstructed images at higher orders of moment and lower values of mean square reconstruction error ε. This is demonstrated by reconstructing the Lena image of resolution 256×256 from the CHFMs. The reconstructed images and the reconstruction error ε are depicted in Fig. 4. Algorithm B and Algorithm C gives the same results, because both are the recursive formulation of the radial functions. It is observed that the direct method becomes numerically unstable after a certain order of moments, while the recursive method remains stable for very high values of p_{max} and q_{max}. The direct method becomes numerically unstable at much lower values of p_{max} and q_{max} than the proposed recursive method. The average reconstruction error ε as a function of p_{max} and q_{max} (for convenience, $p_{max} = q_{max}$) is plotted in Fig. 5 for image size 256×256. It is observed that the average reconstruction error decreases initially when p_{max} and q_{max} increase. Both for the direct and recursive methods ε decreases until $p_{max} = q_{max}$ reaches a value at which ε is minimum. After that, ε starts increasing showing the numerical instability in CHFMs coefficients.

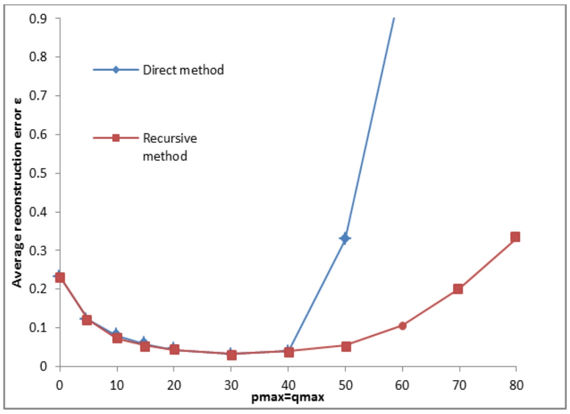


Figure 5. Average reconstruction error ε vs. moment order and repetition, p_{max} and q_{max} for 256×256 pixels image.

V. Conclusion

A fast recursive algorithm for the computation of the CHFMs is presented in this paper. The proposed approach uses recursive relations for the computation of radial and angular functions which are otherwise computation intensive. It is observed that the proposed method reduces the high computation load as compared to the existing direct method which is a great improvement in speed. Therefore, the fast computation of CHFMs would be very useful in many image processing and pattern recognition problems, such as template matching, character recognition, etc., especially in the real time environment.

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